

Linear NMR in the polar phase of ^3He in aerogel

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^3He is an example of the system with non-trivial Cooper pairing. A few different superfluid phases are known in this system. Recently the new one, the polar phase, have been observed in ^3He confined in nematically ordered aerogel [1]. As in other phases, a number of various topological defects can exist in the polar phase. One of them, a half-quantum vortex can couple rotation of the sample and its spin dynamics. The two-dimensional defects, d-solitons, appear if the magnetic field is tilted from the direction of the aerogel strands. These solitons connect half-quantum vortices and can be observed by means of nuclear magnetic resonance (NMR) [2]. Here we present theoretical and numerical studies of linear NMR in the polar phase both in the uniform order-parameter texture and in the presence of half-quantum vortices and d-solitons.

Order-parameter field and energies

We are studying the polar phase of ^3He in nematically ordered aerogel. The order parameter in this system is

$$A_{aj} = \frac{1}{\sqrt{3}} \Delta e^{i\varphi} d_a l_j, \quad (1)$$

where φ is the phase, and \mathbf{d} and \mathbf{l} are unit vectors in spin and orbital spaces respectively. The orbital unit vector \mathbf{l} is directed along the aerogel strands and can not move.

There are three components of the Hamiltonian which are important for spin dynamics: magnetic energy, energy of spin-orbit interaction and gradient energy:

$$\mathcal{H} = F_M + F_{SO} + F_{\nabla}, \quad (2)$$

$$F_M = -(\mathbf{S} \cdot \gamma \mathbf{H}) + \frac{\gamma^2}{2} \chi_{ab}^{-1} S_a S_b, \quad (3)$$

$$F_{SO} = 3g_D \left[A_{jj}^* A_{kk} + A_{jk}^* A_{kj} - \frac{2}{3} A_{jk}^* A_{jk} \right], \quad (4)$$

$$F_{\nabla} = \frac{3}{2} \left[K_1 (\nabla_j A_{ak}^*) (\nabla_j A_{ak}) + K_2 (\nabla_j A_{ak}^*) (\nabla_k A_{aj}) + K_3 (\nabla_j A_{aj}^*) (\nabla_k A_{ak}) \right], \quad (5)$$

where \mathbf{S} is spin and \mathbf{H} is the magnetic field. Susceptibility χ is anisotropic, the axis of anisotropy is \mathbf{d} and minimum of the magnetic energy corresponds to $\mathbf{S} \perp \mathbf{d}$. This can be written as

$$\chi_{ab}^{-1} = \frac{1}{\chi_{\perp}} (\delta_{ab} + \delta d_a d_b), \quad \delta = (\chi_{\perp} - \chi_{\parallel}) / \chi_{\parallel} > 0. \quad (6)$$

Substituting the order parameter (1) into energies and using the fact that \mathbf{l} is uniform we have

$$F_M = -(\mathbf{S} \cdot \gamma \mathbf{H}) + \frac{\gamma^2}{2\chi_{\perp}} [\mathbf{S}^2 + \delta (\mathbf{d} \cdot \mathbf{S})^2], \quad (7)$$

$$F_{SO} = 2\Delta^2 g_D \left[(\mathbf{d} \cdot \mathbf{l})^2 - \frac{1}{3} \right], \quad (8)$$

$$F_{\nabla} = \frac{\Delta^2}{2} K_{jk} [(\nabla_j \varphi)(\nabla_k \varphi) + (\nabla_j d_a)(\nabla_k d_a)], \quad (9)$$

where symmetric matrix $K_{jk} = K_1 \delta_{jk} + (K_2 + K_3) l_j l_k$ is introduced. Motion of the phase φ (sound) is not coupled with the motion of \mathbf{d} (spin waves). Terms with the phase gradients give only a constant contribution to the energy and can be skipped.

Equilibrium texture

Let's first study the static picture. In the equilibrium $\partial \mathcal{H} / \partial S_a = 0$. This means

$$\mathbf{S}^0 + \delta (\mathbf{d}^0 \cdot \mathbf{S}^0) \mathbf{d}^0 = \frac{\chi_{\perp}}{\gamma} \mathbf{H}, \quad (10)$$

where \mathbf{S}^0 and \mathbf{d}^0 are equilibrium values of \mathbf{S} and \mathbf{d} . Multiplying this by \mathbf{d}^0 we can find $(\mathbf{d}^0 \cdot \mathbf{S}^0) = \chi_{\parallel} / \gamma (\mathbf{d}^0 \cdot \mathbf{H})$. then substituting it back to (10) we find the value for the spin in the equilibrium:

$$\gamma S_a^0 = \left[\chi_{\perp} \delta_{ab} - (\chi_{\perp} - \chi_{\parallel}) d_a^0 d_b^0 \right] H_b = \chi_{ab} H_b \quad (11)$$

For calculation of the equilibrium distribution (texture) of the \mathbf{d} vector we will use a coordinate system where $\mathbf{H} \parallel \hat{\mathbf{z}}$ and \mathbf{l} is in $\hat{\mathbf{z}} - \hat{\mathbf{y}}$ plane (See Fig.1). This can be written as

$$\begin{aligned} \mathbf{H} &= \hat{\mathbf{z}} H, & \mathbf{l} &= \hat{\mathbf{y}} \sin \mu + \hat{\mathbf{z}} \cos \mu, \\ \mathbf{d}^0 &= (\hat{\mathbf{x}} \cos \alpha + \hat{\mathbf{y}} \sin \alpha) \sin \beta + \hat{\mathbf{z}} \cos \beta. \end{aligned} \quad (12)$$

Here μ is angle between \mathbf{l} and magnetic field, it is set by the experimental setup because direction of \mathbf{l} is determined by aerogel; β is angle between \mathbf{d}^0 and the field; α

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is azimuthal angle of the \mathbf{d} in the plane, perpendicular to the magnetic field, it is counted from the line, perpendicular to both \mathbf{H} and \mathbf{l} which corresponds to the minimum of energy.

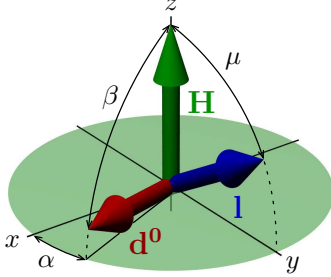


Fig. 1. Angles, used in the texture calculations

The energies (7)-(9) (without constant terms) are:

$$F_M = \frac{1}{2}(\chi_{\perp} - \chi_{\parallel})H^2 \cos^2 \beta, \quad (13)$$

$$F_{SO} = 2g_D \Delta^2 (\sin \alpha \sin \beta \sin \mu + \cos \beta \cos \mu)^2, \quad (14)$$

$$F_{\nabla} = \frac{\Delta^2}{2} K_{jk} [\sin^2 \beta (\nabla_j \alpha)(\nabla_k \alpha) + (\nabla_j \beta)(\nabla_k \beta)] \quad (15)$$

There are two scales introduced by these energies. Ratio of magnetic and gradient energies gives the magnetic length ξ_H and ratio of spin-orbit and gradient energies gives the dipolar length ξ_D . Since the gradient energy is anisotropic, we have different values in directions perpendicular and parallel to the \mathbf{l} vector:

$$\xi_{Hjk}^2 = \frac{K_{jk} \Delta^2}{H^2 (\chi_{\perp} - \chi_{\parallel})}, \quad \xi_{Djk}^2 = \frac{K_{jk}}{4g_D} \quad (16)$$

In the high-field limit $\xi_D \gg \xi_H$. Magnetic energy is in the minimum everywhere excluding small regions of the ξ_H size (for example cores of spin vortices). The small volume of this regions makes them invisible in NMR experiments. In the rest of the volume $\beta = \pi/2$, only variations of α are important and the energy is:

$$\mathcal{H} = \frac{1}{2} K_{jk} \Delta^2 (\nabla_j \alpha)(\nabla_k \alpha) + 2g_D \Delta^2 \sin^2 \alpha \sin^2 \mu \quad (17)$$

The equilibrium state corresponds to the minimum: $\delta \mathcal{H} / \delta \alpha = 0$. Since the energy depends on the gradient we have to use variational derivative

$$\frac{\delta \mathcal{H}}{\delta \alpha} = \frac{\partial \mathcal{H}}{\partial \alpha} - \nabla_j \frac{\partial \mathcal{H}}{\partial \nabla_j \alpha}. \quad (18)$$

Using this for energy (17) we have a simple equation for the distribution of α :

$$2\bar{\xi}_{Djk}^2 \nabla_j \nabla_k \alpha = \sin 2\alpha, \quad (19)$$

$$\text{where } \bar{\xi}_{Djk}^2 = \frac{\xi_{Djk}}{\sin^2 \mu}.$$

One can see that in the case of $\mathbf{H} \parallel \mathbf{l}$ (or $\mu = 0$) there is no length scale in this problem. \mathbf{d} can freely move in the plane perpendicular to the field and only the gradient term is important. Tilting the magnetic field from the \mathbf{l} direction makes the ξ_D finite. At $\mathbf{H} \perp \mathbf{l}$ the length scale reaches its minimum value, ξ_D .

Textural defects

Equation (19) shows that in a tilted magnetic field there are two possible uniform textures with $\alpha = 0$ and $\alpha = \pi$. Vector \mathbf{d} is oriented perpendicularly to both \mathbf{H} and \mathbf{l} and can point in two possible directions. Between this two states there is a *d-soliton*. One can also imagine a *spin vortex* in which vector \mathbf{d} rotates by 2π around the vortex line. Two *d-solitons* should end at this vortex. Looking at the order parameter formula (1) one can see that there can be also a *half-quantum vortex*, in which both vector \mathbf{d} and phase ϕ rotate by π around the vortex line. This is possible because $A_{\alpha j}(\mathbf{d}, \phi) = A_{\alpha j}(-\mathbf{d}, \phi + \pi)$. In the tilted magnetic field one *d-soliton* should end at the vortex. On Fig. 2 two types of vortices are shown.

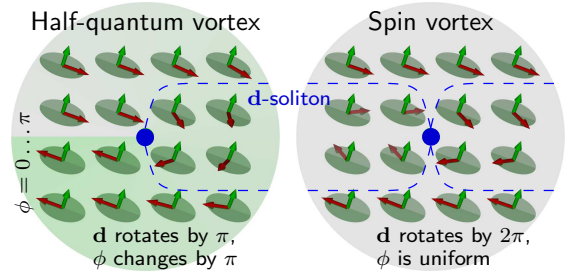


Fig. 2. The half-quantum vortex and the spin vortex in the polar phase of ^3He . Vector \mathbf{l} is perpendicular to the picture plane. Angle $\alpha = 0$ is changing by π between upper and lower parts of the picture. This can be done via either a *d-soliton* or a π jump in the phase (which is shown by color gradient).

The form of the infinite *d-soliton* can be found analytically. In this one-dimensional problem equation (19) has a form of static sine-Gordon equation:

$$\bar{\xi}_D^2 \alpha''(x) = \frac{1}{2} \sin 2\alpha(x), \quad (20)$$

where x is a coordinate perpendicular to the wall. Here the value of $\bar{\xi}_D$ depends on the wall orientation: if x

coordinate goes perpendicular or parallel to \mathbf{l} , it should be $\bar{\xi}_{D\perp}$ or $\bar{\xi}_{D\parallel}$ respectively.

The analytical solution can be obtained by multiplying the equation by a' and integrating with proper boundary conditions. Then for a single soliton with $\sin\alpha(\pm\infty) = 0$ and $\alpha'(\pm\infty) = 0$ we have

$$\bar{\xi}_D^2 (\alpha')^2 = \sin^2 \alpha, \quad (21)$$

and then for the domain wall at $x = 0$:

$$\alpha(x) = 2 \arctan(\exp(x/\bar{\xi}_D)) \quad (22)$$

In the 2D case with isotropic ξ_D (which takes place when the texture is uniform along \mathbf{l} -direction) the sine-Gordon equation has analytic solutions for a number of configurations with spin vortices and solitons [3, 4]. This includes, in particular, the kink on soliton, which represents the 2π spin vortex with two π -solitons being on the opposite sides of it (see right part of Fig. 2). The linear chain of the alternating 2π and -2π vortices (kinks on straight soliton) has also analytic solution. The configuration with two solitons crossing each other may also represent the spin vortex, if each soliton has a kink and the positions of two kinks coincide. This is 4π spin vortex, from which four π -solitons emerge. Such analytic solutions do not take into account the pinning of vortices which exists in the real system.

Spin dynamics

To study spin dynamics we write Hamilton equation using Poisson brackets. Motion of any value A in this approach is given by $\dot{A} = \{\mathcal{H}, A\}$. Choice of coordinates is quite arbitrary as far as we know Poisson brackets for them. Brackets can be found from microscopic considerations, from commutation rules in quantum mechanic, or from symmetry [5]. For spin \mathbf{S} and a vector \mathbf{d} in the spin space the Poisson brackets are

$$\{S_a, S_b\} = -e_{abc} S_c, \quad \{d_a, d_b\} = 0, \quad (23)$$

$$\{d_a, S_b\} = \{S_a, d_b\} = -e_{abc} d_c,$$

and equations of motion:

$$\dot{S}_a = \{\mathcal{H}, S_a\} = \frac{\delta\mathcal{H}}{\delta S_b} \{S_b, S_a\} + \frac{\delta\mathcal{H}}{\delta d_b} \{d_b, S_a\} \quad (24)$$

$$= \frac{\delta\mathcal{H}}{\delta \mathbf{S}} \times \mathbf{S} + \frac{\delta\mathcal{H}}{\delta \mathbf{d}} \times \mathbf{d},$$

$$\dot{d}_a = \{\mathcal{H}, d_a\} = \frac{\delta\mathcal{H}}{\delta S_b} \{S_b, d_a\} + \frac{\delta\mathcal{H}}{\delta d_b} \{d_b, d_a\} \quad (25)$$

$$= \frac{\delta\mathcal{H}}{\delta \mathbf{S}} \times \mathbf{d}.$$

Using these equations one can show that $\frac{d}{dt}(\mathbf{d} \cdot \mathbf{S}) = 0$ and thus the value $(\mathbf{d} \cdot \mathbf{S})$ is an integral of motion.

Derivatives of the Hamiltonian are:

$$\frac{\delta\mathcal{H}}{\delta S_a} = -\gamma H_a + \frac{\gamma^2}{\chi_\perp} [S_a + \delta (\mathbf{d} \cdot \mathbf{S}) d_a], \quad (26)$$

$$\frac{\delta\mathcal{H}}{\delta d_a} = \frac{\delta \gamma^2}{\chi_\perp} (\mathbf{d} \cdot \mathbf{S}) S_a + 4g_D \Delta^2 (\mathbf{d} \cdot \mathbf{l}) l_a - K_{jk} \Delta^2 (\nabla_j \nabla_k d_a). \quad (27)$$

Substituting (26), (27), and (23) into equations (24-25) one has:

$$\dot{\mathbf{S}} = [\mathbf{S} \times \gamma \mathbf{H}] + 4g_D \Delta^2 (\mathbf{d} \cdot \mathbf{l}) [\mathbf{l} \times \mathbf{d}] - K_{jk} \Delta^2 [\nabla_j \nabla_k \mathbf{d} \times \mathbf{d}], \quad (28)$$

$$\dot{\mathbf{d}} = \gamma \left[\mathbf{d} \times \left(\mathbf{H} - \frac{\gamma}{\chi_\perp} \mathbf{S} \right) \right]. \quad (29)$$

Note that the anisotropy of susceptibility do not affect dynamics.

Linearized dynamics

Consider small oscillations near the equilibrium:

$$\mathbf{S} = \mathbf{S}^0 + \delta\mathbf{S}(t), \quad \mathbf{d} = \mathbf{d}^0 + \delta\mathbf{d}(t) \quad (30)$$

Linearize equations, differentiate the first one and exclude $\delta\mathbf{d}$. The result can be written as:

$$\delta\ddot{\mathbf{S}}_a = [\delta\dot{\mathbf{S}} \times \gamma \mathbf{H}]_a + \Lambda_{ab} \delta S_b, \quad (31)$$

$$\Lambda_{ab} = \Omega_P^2 [(\mathbf{d}^0 \cdot \mathbf{l})^2 \delta_{ab} - [\mathbf{l} \times \mathbf{d}^0]_a [\mathbf{l} \times \mathbf{d}^0]_b - (\mathbf{d}^0 \cdot \mathbf{l}) d_a^0 l_b^0] + c_{jk}^2 [(\delta_{ab} - d_a^0 d_b^0) \nabla_j \nabla_k - 2d_b^0 (\nabla_j d_a^0) \nabla_k + d_a^0 (\nabla_j \nabla_k d_b^0) - d_b^0 (\nabla_j \nabla_k d_a^0)]$$

where we introduced parameters

$$\Omega_P^2 = 4g_D \frac{\Delta^2 \gamma^2}{\chi_\perp}, \quad c_{jk}^2 = K_{jk} \frac{\Delta^2 \gamma^2}{\chi_\perp} = \Omega_P^2 \xi_{Djk}^2, \quad (32)$$

and use the fact that $c_{jk} = c_{kj}$.

Consider $H \parallel \hat{z}$ and look for a harmonic solution $\delta\mathbf{S} = \mathbf{s} \exp(i\omega t)$. Then the equation can be written as

$$\begin{aligned} -\omega^2 s_x &= \Lambda_{xb} s_b + i\omega_L \omega s_y, \\ -\omega^2 s_y &= \Lambda_{yb} s_b - i\omega_L \omega s_x, \\ -\omega^2 s_z &= \Lambda_{zb} s_b \end{aligned} \quad (33)$$

In high field (comparing with dipolar and gradient effects) motion of the spin is close to a Larmor precession with frequency $\omega \approx \omega_L = \gamma H$ and $\Lambda \ll \omega_L^2$. One can separate equations by putting s_y from the second equation to the first one and vice versa and neglecting

small terms. We get the same equations for s_x and s_y . This can be written as a single equation for a complex coordinate $s_+ = (s_x + is_y)/\sqrt{2}$:

$$(\omega_L^2 - \omega^2)s_+ = i(\Lambda_{xy} - \Lambda_{yx})s_+ + (\Lambda_{xx} + \Lambda_{yy})s_+ \quad (34)$$

In high field \mathbf{d}^0 is perpendicular to the field and we can use angles (12) with $\beta_n = \pi/2$. Then

$$\begin{aligned} \Lambda_{xx} + \Lambda_{yy} &= \Omega_P^2 [(1 + \sin^2 \alpha) \sin^2 \mu - 1] + c_{jk}^2 \nabla_j \nabla_k \\ \Lambda_{xy} - \Lambda_{yx} &= -\frac{1}{2} \Omega_P^2 \sin 2\alpha \sin^2 \mu \\ &\quad + 2c_{jk}^2 [(\nabla_j \nabla_k \alpha) + (\nabla_j \alpha) \nabla_k]. \end{aligned} \quad (35)$$

Substituting this into (34) and using (19) we have

$$\begin{aligned} (\omega^2 - \omega_L^2)s_+ &= \Omega_P^2 \left\{ \cos^2 \mu - \sin^2 \alpha \sin^2 \mu \right\} s_+ \\ &\quad - c_{jk}^2 \left\{ \nabla_j \nabla_k + i[(\nabla_j \nabla_k \alpha) + 2(\nabla_j \alpha) \nabla_k] \right\} s_+. \end{aligned} \quad (36)$$

One can rewrite the equation in the form:

$$\begin{aligned} (\omega^2 - \omega_L^2)s_+ &= \Omega_P^2 \left\{ \cos^2 \mu - \sin^2 \alpha \sin^2 \mu \right\} s_+ \\ &\quad - c_{jk}^2 \left\{ -\left(\frac{\nabla}{i} + \nabla \alpha\right)^2 + (\nabla \alpha)_{jk}^2 \right\} s_+. \end{aligned} \quad (37)$$

where we use notation $(A)_{jk}^2 = A_j A_k$ (Discussion of Aharonov-Bohm effect, hermitian operators and real/complex solutions.)

It is useful to make a substitution $\bar{s}_+ = s_+ \exp(i\alpha)$. Then the equation contains no imaginary terms:

$$\begin{aligned} (\omega^2 - \omega_L^2)\bar{s}_+ &= \Omega_P^2 \left\{ \cos^2 \mu - \sin^2 \alpha \sin^2 \mu \right\} \bar{s}_+ \\ &\quad - c_{jk}^2 \left\{ \nabla_j \nabla_k + (\nabla_j \alpha)(\nabla_k \alpha) \right\} \bar{s}_+, \end{aligned} \quad (38)$$

The inverse transformation is needed if one need to calculate the actual distribution of magnetization.

NMR in the uniform texture and in the d-soliton

To obtain frequency of the uniform NMR in the uniform texture we put $\alpha = 0$ in (38). Then the frequency is

$$\omega_u = \sqrt{\omega_L^2 + \Omega_P^2 \cos^2 \mu}. \quad (39)$$

This formula can be used to measure Ω_P .

To find the spin wave, localized in the infinite d-soliton we use (38) and the soliton equation (22) for the distribution of α . This gives us

$$\bar{s}_+ = \cosh^{-1}(x/\bar{\xi}_D), \quad (40)$$

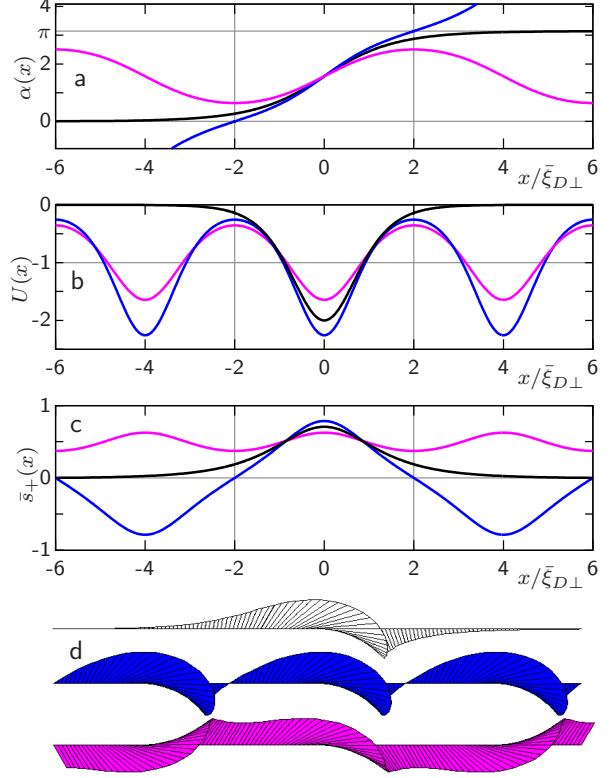


Fig. 3. An example of the calculated texture and the spin wave in 1D soliton structures. Black curves correspond to an infinite soliton, blue and purple ones correspond to periodic structures with same and alternating soliton orientations. The distance between solitons is $D = 4\xi_D$. (a) Texture $\alpha(x)$. (b) Potential for a real-value wave \bar{s}_+ . Energy levels for all three textures are the same, $\lambda = -1$. (c) The real-value wave \bar{s}_+ . Note, that phase of the actual magnetization s_+ rotates by π across each soliton in the direction determined by the soliton orientation. One can check that the total magnetization $|\int s_+ dx|$ for both waves is non-zero. (d) Distribution of the amplitude and phase of the actual magnetization $s_+ = \bar{s}_+ \exp(-i\alpha)$.

where as in (22) the value of ξ_D depends on the domain wall orientation. The frequency is

$$\omega_s = \sqrt{\omega_L^2 + \Omega_P^2 \cos^2 2\mu}. \quad (41)$$

On NMR experiments two peaks are observed, one from the uniform texture and another from the state localized in solitons. The difference between peaks is

$$\delta\omega \approx \frac{\Omega_P^2}{2\omega} \sin^2 \mu \quad (42)$$

An important parameter of the wave is the integral ratio

$$I^M = \frac{|\int_V s_+|^2}{\int_V |s_+|^2}. \quad (43)$$

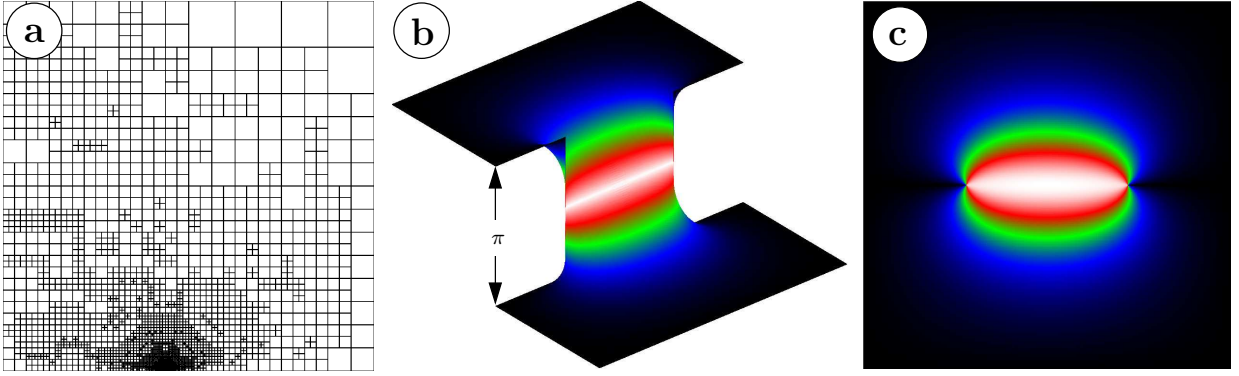


Fig. 4. An example of the calculated texture and the spin wave in the soliton between two half-quantum vortices. **(a)** The calculation grid made of 4696 cells covers one-fourth of the whole area $(8 \times 8) \xi_D$ with two vortices separated by $D = 7 \xi_D$. Density of the grid is chosen according with gradients of the texture, it is higher near vortices. **(b)** Calculated value of α . One can see a smooth rotation by π in the soliton between vortices and π jump on the other side of vortices where phase also changes by π . **(c)** The calculated real-value wave \bar{s}_+ .

It connects the total transverse magnetization M_\perp , measured in NMR experiments and energy E stored in the wave: $M_\perp^2 = 2\chi_\perp I^M E$. For the infinite soliton with length L ($L \gg \xi_D$) the ratio is $I^M = 2L$.

Numerical study of soliton structures

It is interesting to study how various effects can change the frequency of the wave in the soliton. We will do it numerically in one and two-dimensional cases, with solitons perpendicular to the l vector. Using coordinates in units of $\xi_{D\perp}^2$ one can write the equation (19) for the texture as

$$\nabla^2 \alpha = \frac{1}{2} \sin 2\alpha, \quad (44)$$

and equation (38) for the real-value waves as:

$$-\nabla^2 \bar{s}_+ + U(x) \bar{s}_+ = \lambda \bar{s}_+, \quad (45)$$

where potential $U(x) = -(\nabla \alpha)^2 - \sin^2 \alpha$ and

$$\lambda = \frac{\omega^2 - \omega_L^2 - \Omega_P^2 \cos^2 \mu}{\Omega_P^2 \sin^2 \mu} = -\frac{\omega^2 - \omega_u^2}{\omega_s^2 - \omega_u^2}. \quad (46)$$

In the case of an infinite soliton $\omega = \omega_s$ and $\lambda = -1$.

Using the equation (44) we can numerically calculate distribution of α . Then, using equation (45) we can calculate eigenvalues λ .

First consider a 1D periodic structure of parallel solitons, located at some distance D from each other. Solitons have an orientation (direction of $\nabla \alpha$), and two simplest structures which we study are sequences of solitons with same and alternating orientations.

The solution for this problem is shown on Fig. 3. Parameter λ for both periodic structures has the same value -1 as for the infinite soliton.

Let's also study an effect of a finite-length soliton. Consider a two-dimensional problem with two half-quantum vortices parallel to the l vector. Distance between vortices is D . The same equations (44) and (45) are solved numerically in 2D space using *deal.II* library [7]. The code is available in [8]. An example of the calculation is presented on Fig. 4.

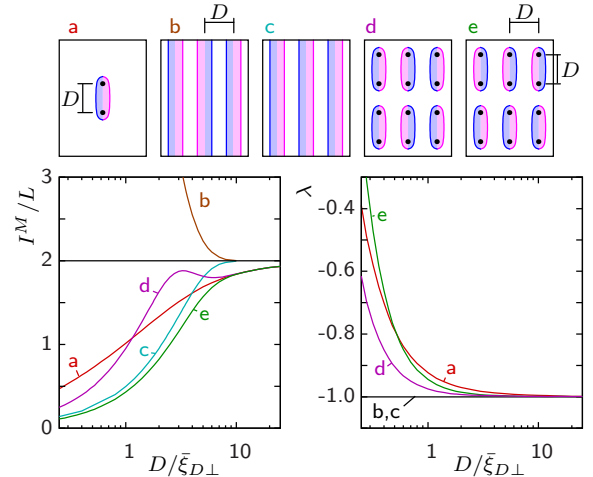


Fig. 5. Calculated values of λ and I^M/L for various soliton structures.

Near a half-quantum vortex, at a distance much smaller than $\xi_{D\perp}$, the textural angle $\alpha \approx \varphi/2 + \text{const.}$, where φ is the azimuthal coordinate. One can see that the potential in (45) is $U(x) \approx (\nabla \alpha)^2 \approx 1/4r^2$ (where r

is distance from the vortex core). The real-value wave \bar{s}_+ can not fall into this hole because of Aharonov-Bohm effect: it should be zero along some radial direction to allow a smooth s_+ distribution. The symmetry reasons tell, that in the case of two vortices with a soliton the wave is zero on the line connecting vortices outside them. The corresponding solution of the wave equation is $\bar{s}_+ \approx \cos(\phi/2 + \text{const.})$, this kind of discontinuity is clearly seen on the calculated wave near vortices.

On Fig. 5 calculated values of λ and I^M/L (where L is the soliton length) are plotted as a function of some structure dimension $D/\bar{\xi}_{D\perp}^2$. There are five structures which are shown on the upper part of the figure: a single soliton with a finite length D ; A periodic structures of infinite solitons with the period D and same or alternating soliton orientations; the combination of both effects, periodic structures of finite solitons with equal length and period (this corresponds to a square lattice of vortices). For large D all curves come to the values for a single infinite soliton: $\lambda = -1$, $I^M = 2L$. The numerical studies show that noticeable deviation of λ from the asymptotic value appears only at high vortex densities, when the inter-vortex distance D is less then a few $\bar{\xi}_D$.

Acknowledgements

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1. Polar phase of superfluid ^3He in anisotropic aerogel. V.V. Dmitriev, A.A. Senin, A.A. Soldatov, A.N. Yudin, *Phys. Rev. Lett.*, **115**, 165304 (2015), arXiv:1507.04275
 2. Observation of half-quantum vortices in superfluid ^3He . S. Autti, V.V. Dmitriev, V.B. Eltsov, J. Mäkinen, G.E. Volovik, A.N. Yudin, V.V. Zavjalov, arXiv:1508.02197
 3. On vortex configurations in two-dimensional sine-Gordon systems with applications to phase transitions of the Kosterlitz-Thouless type and to Josephson junctions. O. Hudak, *Phys. Lett.* **89A**, 245–248 (1982).
 4. Relation Between Certain Quasi-Vortex Solutions and Solitons of the Sine-Gordon Equation and Other Non-linear Equations. A. Nakamura, *J. Phys. Soc. Jpn.* **52**, 1918–1920 (1983).
 5. Poisson brackets in condensed matter physics. I.E. Dzyaloshinskii and G.E. Volovick, *Annals of Physics*, **125**, 67–97 (1980)
 6. Satellite magnetic resonances of a bound pair of half-quantum vortices in rotating superfluid ^3He -A. Chia-Ren Hu, Kazumi Maki, *Phys. Rev. B*, **36**, 6871–6880 (1987)
 7. <https://www.dealii.org>
 8. https://github.com/slazav/dealii_progs